Financial Economics: Time Value of Money and DCF Analysis

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Outline

Outline

Time Value of Money (TVM)

\$20 today is worth more than the expectation of \$20 tomorrow because:

- a bank would pay interest on the \$20
- inflation makes tomorrows \$20 less valuable than today's
- uncertainty of receiving tomorrow's \$20

Outline

Compounding

- Assume that the interest rate is 10%
- What this means is that if you invest \$1 for one year, you have been promised \$1*(1+10/100) or \$1.10 next year
- Investing \$1 for yet another year promises to produce 1.10 *(1+10/100) or \$1.21 in 2-years
- Continuing in this manner you will find that the following amounts will be earned:

1 Year	\$1.1
2 Years	\$1.21
3 Years	\$1.331
4 Years	\$1.4641

Value of Investing $5

More generally, with an investment of $5 at 10% we obtain

1 Year	$5*(1+0.10)	$5.5
2 years	$5.5*(1+0.10)	$6.05
3 years	$6.05*(1+0.10)	$6.655
4 Years	$6.655*(1+0.10)	$7.3205

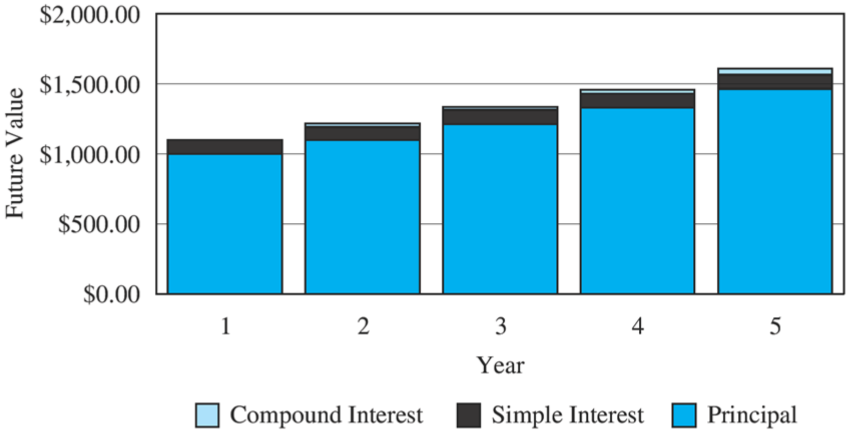
Generalizing the method

Generalizing the method requires some definitions. Let

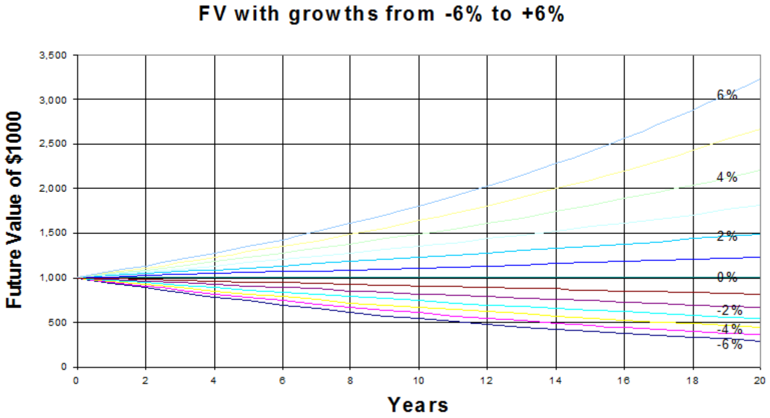
- i be the interest rate
- n be the life of the lump sum investment
- PV be the present value
- FV be the future value

$$FV = PV * (1 + i)^n$$

Future Value and Compound Interest



Future Value of a Lump Sum



Example: Future Value of a Lump Sum

- Your bank offers a CD with an interest rate of 3% for a 5 year investment.
- You wish to invest $1,500 for 5 years, how much will your investment be worth?

$$\begin{aligned} FV &= PV * (1+i)^n \\ &= 1500 * (1+0.03)^5 \\ &= \$1738.9111145 \end{aligned}$$

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$$
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&= 1500 * (1+0.03)^5 \\
&= \$1738.9111145
\end{aligned}
$$

RULE OF 72

Important Reminders:

- This rule says that the number of years it takes for a sum of money to double in value (the doubling time) is approximately equal to the number 72 divided by the interest rate expressed in percent per year
  - Doubling Time = 72/(interest rate)
  - For example, interest rate=5%, doubling time=72/5=14.4 years

Present Value of a Lump Sum

$$\begin{aligned} FV &= PV * (1+i)^n \\ PV &= \frac{FV}{(1+i)^n} \end{aligned}$$

Example: You have been offered \$40,000 for your printing business, payable in 2 years. Given the risk, you require a return of 8%. What is the present value of the offer?

$$\begin{aligned} PV &= \frac{FV}{(1+i)^n} \\ &= \frac{40,000}{(1+0.08)^2} \\ &= \$34293.55281 \end{aligned}$$

Discounting the Future

- Present value is sometimes referred to as “present discounted value.”
- The further in the future a payment is to be received, the smaller its present value
- The higher the interest rate used to discount future payments, the smaller the present value of the payments
- The present value of a series of future payment is simply the sum of the discounted value of each individual payment.

Discounting the Future

Table 3.1 Time, the Interest Rate, and the Present Value of a Payment

Interest Rate	Present Value of a $1,000 payment to be received in . . . 1 Year	5 Years	15 Years	30 Years
1%	$990.10	$951.47	$861.35	$741.92
2%	980.39	905.73	743.01	552.07
5%	952.38	783.53	481.02	231.38
10%	909.09	620.92	239.39	57.31
20%	833.33	401.88	64.91	4.21

Lump Sums Formula

You have solved a present value and a future value of a lump sum. There remains two other variables that may be solved for

- interest, i
- number of periods, n

$$
\begin{aligned}
FV &= PV * (1+i)^n \\
\frac{FV}{PV} &= (1+i)^n \\
(1+i) &= \sqrt[n]{\frac{FV}{PV}} \\
i &= \sqrt[n]{\frac{FV}{PV}} - 1
\end{aligned}
$$

Example: Interest Rate on a Lump Sum Investment

If you invest $15,000 for ten years, you receive $30,000. What is your annual return?

$$
\begin{aligned}
i &= \sqrt[n]{\frac{FV}{PV}} - 1 \\
&= \sqrt[n]{\frac{30000}{15000}} - 1 \\
&\simeq 0.07177 \\
&= 7.18\%(\textit{to the nearest basis point})
\end{aligned}
$$

Review of Logarithms

- The basic properties of logarithms that are used by finance are:

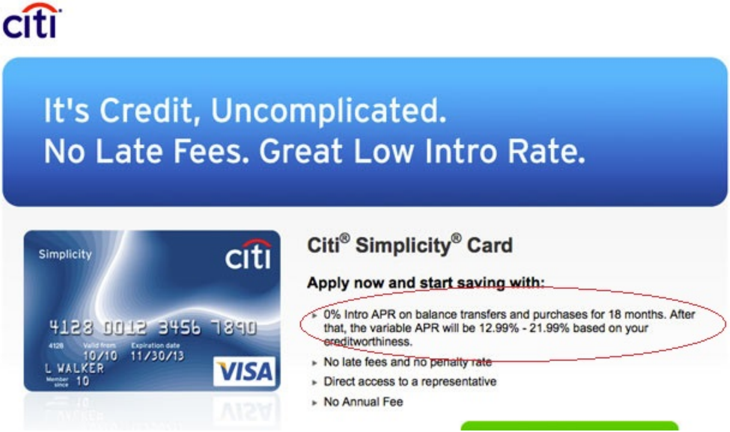
$$
\begin{aligned}
e^{ln(x)} &= x, \quad x > 0 \\
ln(e^x) &= x \\
ln(x * y) &= ln(x) + ln(y) \\
ln(x^y) &= y ln(x)
\end{aligned}
$$

Solving Lump Sum Cash Flow for Number of Periods

$$
\begin{aligned}
FV &= PV * (1+i)^n \\
\frac{FV}{PV} &= (1+i)^n \\
ln(\frac{FV}{PV}) &= ln((1+i)^n) = n * ln(1+i) \\
n &= \frac{ln(\frac{FV}{PV})}{ln(1+i)} = \frac{ln(FV) - ln(PV)}{ln(1+i)}
\end{aligned}
$$

Outline

annual percentage rate (APR)



The Frequency of Compounding

- You have a credit card that carries a rate of interest of 18% per year compounded monthly. What is the interest rate compounded annually?
- That is, if you borrowed \$1 with the card, what would you owe at the end of a year?

The Frequency of Compounding

- 18% per year compounded monthly is just **code** for $18\%/12 = 1.5\%$ per month
- All calculation must be expressed in terms of consistent units
- A raw rate of interest expressed in terms of years and months may never be used in a calculation
- **The annual rate compounded monthly is code for one twelfth of the stated rate per month compounded monthly**
- The year is the macroperiod, and the month is the microperiod
- In this case there are 12 microperiods in one macroperiod
- When a rate is expressed in terms of a macroperiod compounded with a different microperiod, then it is a nominal or annual percentage rate (APR)
- If macroperiod = microperiod then the rate is referred to as a the real or effective rate based on that period

The Frequency of Compounding

- Assume m microperiods in a macroperiod and a nominal rate k per *macroperiod* compounded micro-periodically. That is the effective rate is k/m per *microperiod*.
- Invest \$1 for one macroperiod to obtain $\$1 * (1 + k/n)^n$, producing an effective rate over the macroperiod of

$$(\$1 * (1 + k/n)^n - \$1)/\$1 = (1 + k/n)^n - 1$$

Credit Card

- If the credit card pays an APR of 18% per year compounded monthly. The monthly rate is $18\%/12 = 1.5\%$ so the 'real' annual rate is $(1 + 0.015)^{12} - 1 = 19.56\%$
- The two equal APR with different frequency of compounding have different effective annual rates(EFF):

Figure: Effective Annual Rates of an APR of 18%

Annual Percentage rate	Frequency of Compounding	Annual Effective Rate
18	1	18.00
18	2	18.81
18	4	19.25
18	12	19.56
18	52	19.68
18	365	19.72

The Frequency of Compounding

- Note that as the frequency of compounding increases, so does the annual effective rate
- What occurs as the frequency of compounding rises to infinity?

$$EFF = \lim_{m \to \infty} [(1 + \frac{k_m}{m})^m] - 1 = e^{k_\infty} - 1$$

- The effective annual rate thats equivalent to an annual percentage rate of 18% is then $e^{0.18} - 1 = 19.72\%$
- More precision shows that moving from daily compounding to continuous compounding gains 0.53 of one basis point

The Frequency of Compounding

- A bank determines that it needs an effective rate of 12% on car loans to medium risk borrowers
- What annual percentage rates may it offer?

$$
\begin{aligned}
1 + EFF &= (1 + \frac{k_m}{m})^m \\
(1 + \frac{k_m}{m})^m &= (1 + EFF)^{\frac{1}{m}} \\
k_m &= m * [(1 + EFF)^{1/m} - 1]
\end{aligned}
$$

The Frequency of Compounding

Annual Effective Rate	Compounding Frequency	Annual Percentage Rate
12	1	12.00
12	2	11.66
12	4	11.49
12	12	11.39
12	52	11.35
12	365	11.33
12	Infinity	11.33

The Frequency of Compounding

- Many lenders and borrowers do not have a clear understanding of APRs, but institutional lenders and borrowers do
- Institutions are therefore able to extract a few basis points from consumers, but why bother?
- Financial intermediaries profit from differences in the lending and borrowing rates. Overheads, bad loans and competition results in a narrow margin. Small rate gains therefore result in a large increases in institutional profits
- In the long term, ill-informed consumers lose because of compounding

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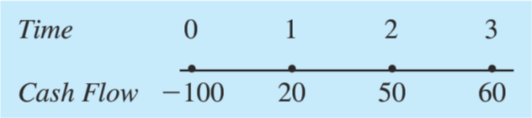
Outline

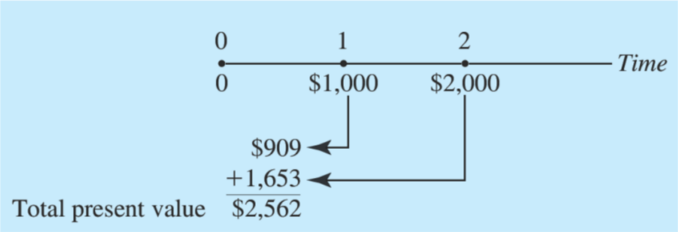
Multiple Cash Flows

- Time Lines
- Future Value of a Stream of Cash Flow
- Present Value of a Stream of Cash Flows
- Investing with Multiple Cash Flows

Figure: Time Line



Present Value of Multiple Cash Flows



Valuing a Contract

Jeremy Lin played the 2011-2012 NBA season with the New York Knicks.
When he became a free agent, the Houston Rockets offered him a contract that would pay him a total of $25 million, which is a backloaded contract offer that would pay him a below-average salary $5 million during the first two years of a three-year, before ballooning to $15 million in the third year of the contract.
What is the true value of the contract?

Discounting and the Prices of Financial Assets

Discounting gives us a way of determining the prices of financial assets. By adding up the present values of all the payments, we have the dollar amount that a buyer will pay for the asset. In other words, we have determined the asset's price.

Outline

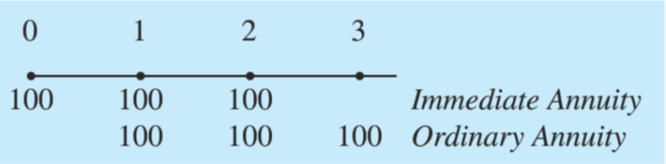
Annuities

Financial analysts use several annuities with differing assumptions about the first payment. We will examine just two:

- regular annuity with its first coupon one period from now, (detail look)
- annuity due with its first coupon today, (cursory look)

Annuity: *a stream of equal payments over equal time intervals.*

Figure: Cash Flow Diagram of Annuities



Rationale for Annuity Formula

- a sequence of equally spaced identical cash flows is a common occurrence, so automation pays off
- a typical annuity is a mortgage which may have 360 monthly payments, a lot of work for using elementary methods

Assumptions Regular Annuity

- the first cash flow will occur exactly one period from now
- all subsequent cash flows are separated by exactly one period
- all periods are of equal length
- **the term structure of interest is flat**
- all cash flows have the same (nominal) value
- the present value of a sum of present values is the sum of the present values

Annuity Formula Notation

- PV = the present value of the annuity
- i = interest rate to be earned over the life of the annuity
- n = the number of payments
- pmt = the periodic payment

Derivation of PV of Annuity Formula

$$PV = \frac{pmt}{(1+i)} + \frac{pmt}{(1+i)^2} + ... + \frac{pmt}{(1+i)^{n-1}} + \frac{pmt}{(1+i)^n}$$

$$PV = pmt \times [\frac{1}{1+i} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}]$$

Derivation of PV of Annuity Formula

$$
\begin{aligned}
& PV \times (1+i) \\
= \quad & pmt \times (1+i)[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}] \\
= \quad & pmt \times [\frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \ldots \\
+ \quad & \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + (\frac{1}{(1+i)^n} - \frac{1}{(1+i)^n})] \\
= \quad & pmt \times \frac{1}{(1+i)^0} + pmt \times [\frac{1}{(1+i)^1} + \ldots \\
+ \quad & \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}] - pmt \times \frac{1}{(1+i)^n} \\
= \quad & pmt \times \frac{1}{(1+i)^0} + PV - pmt \times \frac{1}{(1+i)^n}
\end{aligned}
$$

Derivation of PV of Annuity Formula

$$PV \times (1+i) - PV = pmt - pmt \times \frac{1}{(1+i)^n}$$

$$PV = \frac{pmt \times [1 - \frac{1}{(1+i)^n}]}{i} = \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}]$$

PV of Annuity Formula

$$PV = \frac{pmt \times [1 - \frac{1}{(1+i)^n}]}{i} = \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}]$$

$$payment: \quad pmt = \frac{PV * i}{1 - (1+i)^{-n}}$$

PV Annuity Formula: Number of Payments

$$\begin{aligned}
PV &= \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}] \\
\frac{PV \times i}{pmt} &= 1 - \frac{1}{(1+i)^n} \\
(1+i)^{-n} &= 1 - \frac{PV \times i}{pmt} \\
-n \times ln(1+i) &= ln(1 - \frac{PV \times i}{pmt}) \\
n &= -\frac{ln(1 - \frac{PV \times i}{pmt})}{ln(1+i)}
\end{aligned}$$

Annuity Formula: PV Annuity Due

$$
\begin{aligned}
PV_{due} &= PV_{reg} \times (1+i) \\
&= \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}] \times (1+i) \\
&= \frac{pmt}{i} \times [(1+i) - (1+i)^{1-n}]
\end{aligned}
$$

Derivation of FV of Annuity Formula

$$PV = \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}] \quad (reg.\ annuity)$$

$$FV = PV \times (1+i)^n \quad (lump\ sum)$$

$$FV = \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}] \times (1+i)^n$$

$$= \frac{pmt}{i} \times [(1+i)^n - 1]$$

$$payment: \quad pmt = \frac{FV * i}{(1+i)^n - 1}$$

FV Annuity Formula: Number of Payments

$$
\begin{aligned}
FV &= \frac{pmt}{i} \times [(1+i)^n - 1] \\
1 + \frac{FV * i}{pmt} &= (1+i)^n \\
ln((1+i)^n) &= n * ln(1+i) = ln(1 + \frac{FV * i}{pmt}) \\
n &= \frac{ln(1 + \frac{FV*i}{pmt})}{ln(1+i)}
\end{aligned}
$$

FV Annuity Formula: Return

- **There is no transcendental solution to the PV of an annuity equation in terms of the interest rate.**
- Numerical methods have to be employed

Outline

Perpetual Annuities

- A Perpetuity is with no maturity date that does not repay principal but pays annuities forever
- Recall the annuity formula:

$$PV = \frac{pmt}{i} \times [1 - \frac{1}{(1+i)^n}]$$

- Let $n \to \infty$ with $i > 0$:

$$PV = \frac{pmt}{i}$$

Outline

Loan Amortization: Mortgage

- early repayment permitted at any time during mortgage's 360 monthly payments
- market interest rates may fluctuate, but the loan's rate is a constant 1/2% per month
- the mortgage requires 10% equity (down payment) and "three points" (fee)
- assume a $500,000 house price

Mortgage: The payment

- We will examine this problem using a financial calculator
- The first quantity to determine is the amount of the loan and the points

$$Loan = \$500,000 * (1 - 0.1) = \$450,000$$

$$points = \$500,000 * (1 - 0.1) * 0.03 = \$13,500$$

Calculator Solution

- $PV = -\$450,000$
- $i = 0.5\%$
- $n = 360$
- $FV = 0$
- $pmt = ?$
- result= 2697.87 (monthly repayment)

Calculator Solution

- $PV = -\$450,000$
- $i = 0.5\%$
- $n = 360$
- $FV = 0$
- $pmt = ?$
- result$=$ 2697.87 (monthly repayment)

Mortgage: Early Repayment

- Assume that the family plans to sell the house after exactly 60 payments, what will be the outstanding principle?

Mortgage Repayment: Issues

- The outstanding principle is the present value (at repayment date) of the remaining payments on the mortgage
- There are in this case 360-60 $=$ 300 remaining payments, starting with the one 1-month from now

Calculator Solution

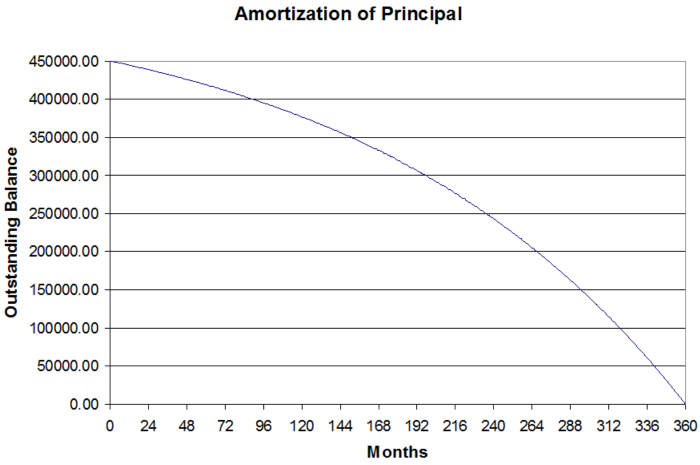
n	i	PV	FV	pmt	result
360	0.5%	-450,000	0	?	2697.98
300	0.5%	?	0	2697.98	-418,745

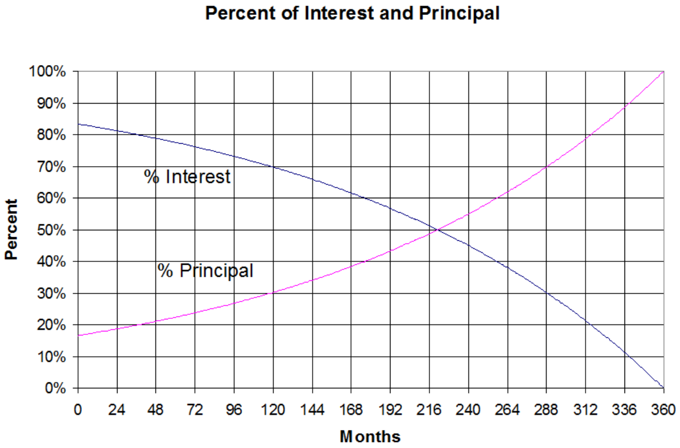
Summary of Payments

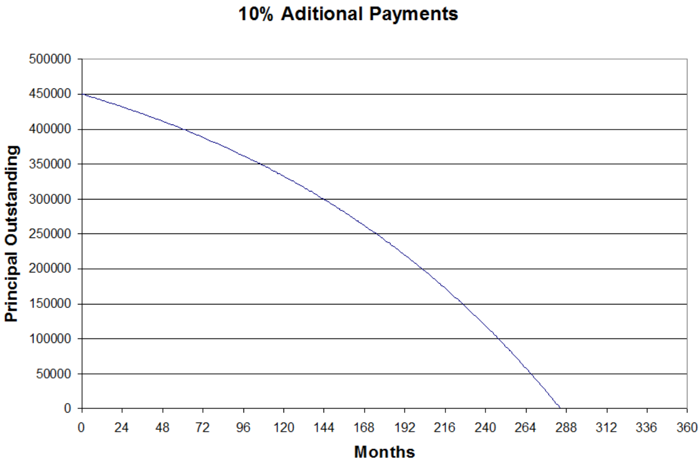
- The family has made 60 payments = $2697.98*60 = $161,878.64
- Their mortgage repayment = 450,000 - 418,744.61 = $31,255.39
- Interest = payments - principle reduction = 161,878.64 - 31,255.39 = $130,623.25

Outstanding Balance as a Function of Time

- The following graphs illustrate that in the early years, monthly payment are mostly interest. In latter years, the payments are mostly principle
- Recall that only the interest portion is tax-deductible, so the tax shelter decays


Amortization of Principal
Outstanding Balance
450000.00
400000.00
350000.00
300000.00
250000.00
200000.00
150000.00
100000.00
50000.00
0.00
0
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
Months


Percent of Interest and Principal
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Percent
% Interest
% Principal
0
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
Months

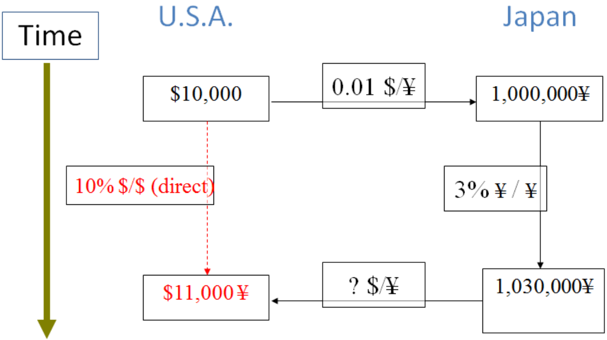

10% Aditional Payments
Principal Outstanding
500000
450000
400000
350000
300000
250000
200000
150000
100000
50000
0
0
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
Months

Outline

Exchange Rates and Time Value of Money

You are considering the choice:

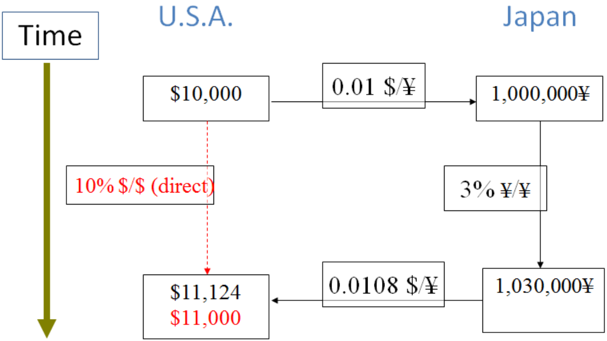
- Investing \$10,000 in dollar-denominated bonds offering 10% / year
- Investing \$10,000 in yen-denominated bonds offering 3% / year; Assume an exchange rate of 0.01


Time
U.S.A.
Japan
$10,000
0.01 $/¥
1,000,000¥
10% $/$ (direct)
3% ¥ / ¥
$11,000 ¥
? $/¥
1,030,000¥

Exchange Rate Diagram

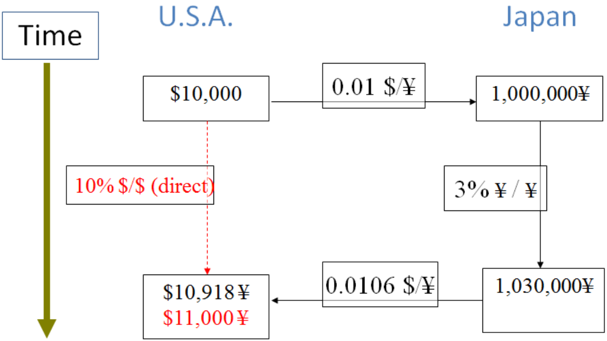
You are considering the choice:

- Review of the diagram indicates that you will end the year with either
  - \$11,000 or
  - ¥1,030,000
- If the \$ price of the yen rises by 8%/year then the year-end exchange rate will be \$0.0108/ ¥


Time
U.S.A.
Japan
$10,000
0.01 $/¥
1,000,000¥
10% $/$ (direct)
3% ¥/¥
$11,124
$11,000
0.0108 $/¥
1,030,000¥

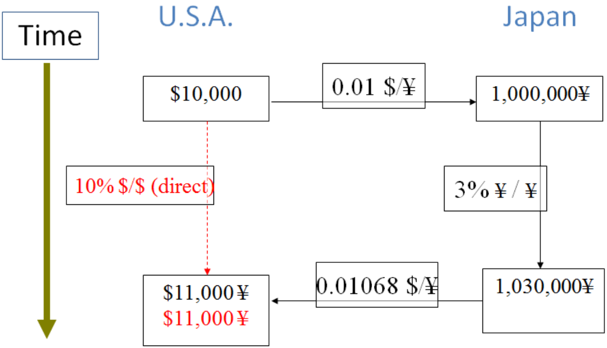
Interpretation and Another Scenario

- In the case of the \$ price of ¥rising by 8% you gain \$124 on your investment
- Now, if the \$ price of ¥rises by 6%, the exchange rate in one year will be \$0.0106


Time
U.S.A.
Japan
$10,000
0.01 $/¥
1,000,000¥
10% $/$ (direct)
3% ¥ / ¥
$10,918 ¥
$11,000 ¥
0.0106 $/¥
1,030,000¥

Interpretation

- In this case, you will lose \$82 by investing in the Japanese bond
- If you divide proceeds of the US investment by those of the Japanese investment, you obtain the exchange rate at which you are indifferent
- $\$11{,}000/¥1{,}030{,}000 = 0.1068\ \$/¥$


Time
U.S.A.
Japan
$10,000
0.01 $/¥
1,000,000¥
10% $/$ (direct)
3% ¥ / ¥
$11,000¥
$11,000¥
0.01068 $/¥
1,030,000¥

Conclusion

- If the yen price actually rises by more than 6.8% during the coming year then the yen bond is a better investment

Financial Decision in an International Context

- International currency investors borrow and lend in
  - Their own currency
  - The currency of countries with which they do business but wish to hedge
  - Currencies that appear to offer a better deal
- Exchange rate fluctuations can result in unexpected gains and losses

Computing NPV in Different Currencies

In any time-value-of-money calculation, the cash flows and interest rates must be denominated in the same currency

- USA project U requires an investment of \$10,000, as does a Japanese project J. U generates \$6,000/year for 5 years, and project J generates ¥575,000/year for 5 years
- The US interest is 6%, the Japanese interest is 4%, and the current exchange rate is 0.01

Solution

- Determine the present value of U in \$ by discounting the 5 payments at 6%, and subtract the initial investment of \$10,000
- Determine the present value of J in ¥by discounting the 5 payments at 4%, and subtract the initial investment of ¥1,000,000
- Obtain \$15,274 & ¥1,5599,798 respectively
- Convert the ¥1,5599,798 to \$ using the current exchange rate to obtain \$15,600
- The Japanese NPV of ¥of \$15,600 is higher than the USA NPV or \$15,274, so invest in the Japanese project

Outline

Inflation and Discounted Cash Flow Analysis

We will use the notation

- i_n the rate of interest in nominal terms
- i_r the rate of interest in real terms
- r the rate of inflation

- From chapter 2 we have the relationship

$$1 + i_r = \frac{1 + i_n}{1 + r} \Leftrightarrow i_r = \frac{i_n - r}{1 + r}$$

Illustration

What is the real rate of interest if the nominal rate is 8% and inflation is 5%?

$$1 + i_r = \frac{1 + i_n}{1 + r} \Leftrightarrow i_r = \frac{i_n - r}{1 + r}$$

$$i_r = \frac{0.08 - 0.05}{1.05} = 0.0286 = 2.86\%$$

- The real rate or return determines the spending power of your savings
- The nominal value of your wealth is only as important as its purchasing power

Investing in Inflation-protected CDs

You have decided to invest \$10,000 for the next 12-months. You are offered two choices

- A nominal CD paying a 8% return
- A real CD paying 3% + inflation rate

If you anticipate the inflation being

- Below 5% invest in the nominal security
- Above 5% invest in the real security
- Equal to 5% invest in either

Why Debtors Gain From Unanticipated Inflation

You borrow $10,000 at 8% interest. The todays spending power of the repayment is $10,000*1.08/ (1+inflation)

- If the actual inflation is the expected 6%, then the real cost of the loan in todays money is $10,188.68
- If the actual inflation is 10%, then the loans real cost (in todays values) is $9,818.18

Unexpected inflation benefits borrower

Inflation and Present Value

- A common planning situation is determining how long it takes to save for something
- The problem is that the thing being saved for increases in (nominal) price due to inflation
- Using a real approach solves this issue

Inflation and Present Value

Illustration:

- Assume that a boat costs $20,000 today
- General inflation is expected to be 3%
- At today's values, you can save at an inflation adjusted rate of $3,000/year, making the first deposit 1-year hence
- You are able to earn 12% loans at Honest Joes Pawn Emporium

When is the boat yours?

Boat Illustration Continued

Solution:

- The boat is already at nominal value
- To convert the nominal rate to the real rate

$$\begin{aligned} I_{real} &= (I_{nominal} - inflation)/(1 + inflation) \\ &= (0.12 - 0.03)/1.03 = 8.7378641\% \end{aligned}$$

- Using your calculator

$$N \rightarrow ?;\ \ I \rightarrow 8.7378641;\ \ FV \rightarrow 0;$$
$$PMT \rightarrow 3000;\ \ PV \rightarrow 20000$$

- Result: $n = 5.48$ *years*, (6 years w/ change)
- Conclusion: Given boater makes deposits at the end of each year, the boat will not be hers for six years