Financial Economics: Risk Aversion and Investment Decisions, Modern Portfolio Theory

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Outline

- The backward induction, three-step solution to modern portfolio theory problems
- Exact and approximate foundations of mean-variance utility functionals
- The normality assumption applied to asset returns
- Building the mean-variance efficient frontier in the two-asset case
- Generalizing the MVF to the N-asset case and to the presence of a riskless asset
- Quadratic Programming construction and the role of constraints
- The separation theorem
Modern Portfolio Theory

We can elaborate on our previous portfolio problem

$$\max_a E u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]$$

by considering \( N > 1 \) risky assets with returns \((\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N)\)

$$\max_{a_1, a_2, \ldots, a_N} E u[Y_0(1 + r_f) + \sum_{i=1}^{N} a_i(\tilde{r}_i - r_f)]$$

$$= \max_{w_1, w_2, \ldots, w_N} E u[Y_0(1 + r_f) + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)]$$

$$= \max_{w_1, w_2, \ldots, w_N} E u[Y_0(1 + \tilde{r}_p)] = Eu(\tilde{Y}_1)$$

where \( w_i = a_i / Y_0 \) is the share of initial wealth allocated to each asset. \( \tilde{r}_p \) is the portfolio rate of return.
Modern Portfolio Theory examines the solution to this problem assuming that investors have mean-variance utility, that is, assuming that investors’ preferences can be represented by a trade-off between the mean (expected value) and variance of the $N$ asset returns.

MPT was developed by Harry Markowitz (US, b.1927, Nobel Prize 1990) in the early 1950s, the classic paper being his article “Portfolio Selection,” Journal of Finance Vol.7 (March 1952): pp.77-91.
Modern Portfolio Theory: three steps

Assume that utility is provided by bundles of consumption goods, $u(c_1, c_2, ..., c_n)$ where the indexing is cross dates and states

- States of nature are mutually exclusive
- For each date and state of nature ($\theta$) there is a traditional budget constraint:

$$p_{1\theta}c_{1\theta} + p_{2\theta}c_{2\theta} + ... + p_{m\theta}c_{m\theta} \leq Y_{\theta}$$

where the indexing runs across goods for a given state $\theta$; the $m$ quantities $c_{i\theta}$; and the $m$ prices $p_{i\theta}$; ($i = 1, 2, ..., m$) correspond to the $m$ goods available in state of nature $\theta$

- $Y_{\theta}$ is the “end of period” wealth level available in that same state
Modern Portfolio Theory: three steps

MPT summarizes an individual’s decision problem as being undertaken sequentially, in three steps.

1) **The Consumption-Savings Decision**: how to split period zero income/wealth \( Y_0 \) between current consumption now \( C_0 \) and saving \( S_0 \) for consumption in the future where \( C_0 + S_0 = Y_0 \)

2) **The Portfolio Problem**: choose assets in which to invest one’s savings so as to obtain a desired pattern of end-of-period wealth across the various states of nature; this means allocating \( (Y_0 − C_0) \) between a risk-free and \( N \) risky assets

3) **The Consumption Choice**: Given the realized state of nature and the wealth level obtained, the choice of consumption bundles to maximize the utility function

\[
Y_\theta = (Y_0 − C_0)[(1 + r_f) + \sum_{i=1}^{N} w_i(r_i\theta - r_f)]
\]
Modern Portfolio Theory: Backward Induction

It is fruitful to work by backward induction, starting from step 3.

- Step 3 is a standard microeconomic problem and its solution can be summarized by a Bernoulli utility function $u(Y_\theta)$ representing the (maximum) level of utility that results from optimizing in step 3 given that the wealth available in state $\theta$ is $Y_\theta$:

$$u(Y_\theta) \equiv \text{def} \max_{(c_{1\theta}, \ldots, c_{m\theta})} u(c_{1\theta}, \ldots, c_{m\theta})$$

s.t. $p_{1\theta}c_{1\theta} + p_{2\theta}c_{2\theta} + \ldots + p_{m\theta}c_{m\theta} \leq Y_\theta$
Modern Portfolio Theory: Backward Induction

- Maximizing $Eu(Y_\theta)$ across all states of nature becomes the objective of step 2:

$$\max_{(w_1, w_2, \ldots, w_N)} Eu(\tilde{Y}) = \sum_{\theta} \pi_\theta u(Y_\theta)$$

The end-of-period wealth can be written as

$$\tilde{Y} = (Y_0 - C_0)(1 + \tilde{r}_P)$$

$$\tilde{r}_P = r_f + \sum_{i=1}^{N} w_i(\tilde{r}_i - r_f)$$
Modern Portfolio Theory: Backward Induction

- Clearly an appropriate redefinition of the utility function leads to

\[
\max Eu(\tilde{Y}) = \max Eu[(Y_0 - C_0)(1 + \tilde{r}_P)] = \text{def} \ \max E\hat{u}(\tilde{r}_P)
\]

The level of investable wealth, \((Y_0 - C_0)\), becomes a parameter of the “U-hat” representation.

- Finally, given the characteristics (e.g., expected return, standard deviation) of the optimally chosen portfolio, the optimal consumption and savings levels can be selected, step 1.

- From now on we work with utility functions defined on \(\tilde{r}_P\).

- This utility index can be further constrained to be a function of the mean and variance of the probability distribution of \(r_P\).

- This simplification can be accepted either as a working approximation or it may result from two further (alternative) hypotheses made within the expected utility framework.
The mean-variance utility hypothesis seemed natural at the time the MPT first appeared, and it retains some intuitive appeal today. But viewed in the context of more recent developments in financial economics, particularly the development of vN-M expected utility theory, it now looks a bit peculiar.

A first question for us, therefore, is: Under what conditions will investors have preferences over the means and variances of asset returns?
Justifying Mean-Variance Utility

In the **exact case**, we have two avenues:

- A decision maker’s utility function is quadratic,
- Asset returns are (jointly) normally distributed,

The main justification for using a mean-variance approximation is its tractability

- Probability distributions are cumbersome to manipulate and difficult to estimate empirically
- Summarizing them by their first two moments is appealing

In the **approximate case**, using a simple Taylor series approximation, one can also see that the mean and variance of an agent’s wealth distribution are critical to the determination of his expected utility for any distribution.
Justifying Mean-Variance Utility

If we start, as we did previously, by assuming an investor has preferences over terminal wealth $\tilde{Y}$, potentially random because of randomness in the asset returns, described by a vN-M expected utility function $E[u(\tilde{Y})]$ we can write

$$\tilde{Y} = E(\tilde{Y}) + [\tilde{Y} - E(\tilde{Y})]$$

and interpret the portfolio problem as a trade-off between the expected payoff

$$E(\tilde{Y})$$

and the size of the “bet”

$$[\tilde{Y} - E(\tilde{Y})]$$
Justifying Mean-Variance Utility

With this interpretation in mind, consider a second-order Taylor approximation of the Bernoulli utility function $u$ once the outcome $[\tilde{Y} - E(\tilde{Y})]$ of the bet is known:

$$u(\tilde{Y}) \approx u[E(\tilde{Y})] + u'[E(\tilde{Y})][\tilde{Y} - E(\tilde{Y})] + \frac{1}{2} u''[E(\tilde{Y})][\tilde{Y} - E(\tilde{Y})]^2$$

Now go back to the beginning of the period, before the outcome of the bet is known, and take expectations to obtain

$$E[u(\tilde{Y})] \approx u[E(\tilde{Y})] + \frac{1}{2} u''[E(\tilde{Y})]\sigma^2(\tilde{Y})$$
Justifying Mean-Variance Utility

\[ E[u(\tilde{Y})] \approx u[E(\tilde{Y})] + \frac{1}{2} u''[E(\tilde{Y})]\sigma^2(\tilde{Y}) \]

The right-hand side of this expression is in the desired form: if \( u \) is increasing, it rewards higher **mean returns** and if \( u \) is concave, it penalizes higher **variance** in returns.

So one possible justification for mean-variance utility is to assume that the size of the portfolio bet \( \tilde{Y} - E(\tilde{Y}) \) is small enough to make this Taylor approximation a good one.

But is it safe to assume that portfolio bets are small?
A second possibility is to assume that the Bernoulli utility function is quadratic, with

\[ u(Y) = a + bY + cY^2 \]

with \( b > 0 \) and \( c < 0 \). Then

\[
\begin{align*}
  u'(Y) &= b + 2cY \\
  u''(Y) &= 2c
\end{align*}
\]

so that \( u'''(Y) = 0 \) and all higher-order derivatives are zero as well. In this case, the second-order Taylor approximation holds exactly.
Quadratic utility function

Note, however, that for a quadratic utility function

$$R_A(Y) = - \frac{u''(Y)}{u'(Y)} = - \frac{2c}{b + 2cY}$$

which is increasing in $Y$.

Hence, quadratic utility has the undesirable implication that the amount of wealth allocated to risky investments declines when wealth increases.
Normality Assumption

There is a result from probability theory: if \( \tilde{Y} \) is **normally distributed** with mean \( \mu_Y = E(\tilde{Y}) \) and standard deviation \( \sigma_Y = (E[\tilde{Y} - E(\tilde{Y})]^2)^{1/2} \) then the expectation of any function of \( \tilde{Y} \) can be written as a function of \( \mu_Y \) and \( \sigma_Y \).

Hence, in particular, there exists a function \( v \) such that

\[
E[u(\tilde{Y})] = v(\mu_Y, \sigma_Y)
\]
Normality Assumption

The result follows from a more basic property of the normal distribution: its location and shape is described completely by its mean and variance.
If $\tilde{Y}$ is normally distributed, there exists a function $v$ such that

$$Eu(\tilde{Y}) = v(\mu_Y, \sigma_Y)$$

Moreover, if $\tilde{Y}$ is normally distributed and

- $u$ is increasing, then $v$ is increasing in $\mu_Y$
- $u$ is concave, then $v$ is decreasing in $\sigma_Y$
- $u$ is concave, then indifference curves defined over $\mu_Y$ and $\sigma_Y$ are convex
Since $\mu_Y$ is a “good” and $\sigma_Y$ is a “bad,” indifference curves slope up. But if $u$ is concave, these indifference curves will still be convex.
Problems with the normality assumption:

- Limited liability instruments such as stocks can pay at worst a negative return of −100% (complete loss of the investment).
- Returns on assets like options are highly non-normal.
- While the normal is perfectly symmetric about its mean, high-frequency returns are frequently skewed to the right and index returns appear skewed to the left.

\[
S(\tilde{r}_{it}) = E\left[\frac{(r_{it} - \mu_i)^3}{\sigma_i^3}\right]
\]

- Sample high-frequency return distributions for many assets exhibit excess kurtosis or “fat tails”.

\[
K(\tilde{r}_{it}) = E\left[\frac{(r_{it} - \mu_i)^4}{\sigma_i^4}\right]
\]
The mean-variance utility hypothesis is intuitively appealing and can be justified with reference to vN-M expected utility theory under various additional assumptions.

Still, it’s important to recognize its limitations: you probably wouldn’t want to use it to design sophisticated investment strategies that involve very large risks or make use of options and you probably wouldn’t want to use it to study how portfolio strategies or risk-taking behavior changes with wealth.
Mean-Variance Dominance

- In a mean-variance (M-V) framework, an investor’s wants to maximize a function $u(\mu_r, \sigma_P)$
- She likes expected return ($\mu_r$) and dislikes standard deviation ($\sigma_P$)
- Recall that portfolio A is said to exhibit mean-variance dominance over portfolio B if either
  
  $\mu_A > \mu_B$ and $\sigma_A \leq \sigma_B$
  
  $\mu_A \geq \mu_B$ and $\sigma_A < \sigma_B$

- We can then define the efficient frontier as the locus of all non-dominated portfolios in the mean-standard deviation space
By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.

The shape of the efficient frontier is of primary interest; let us examine the efficient frontier in the two-asset case for a variety of possible asset return correlations.
Two Risky Assets

One of the most important lessons that we can take from modern portfolio theory involves the gains from diversification.

To see where these gains come from, consider forming a portfolio from two risky assets:

\[ \tilde{r}_1, \tilde{r}_2 = \text{random returns} \]
\[ \mu_1, \mu_2 = \text{expected returns} \]
\[ \sigma_1, \sigma_2 = \text{standard deviations} \]

Assume \( \mu_1 > \mu_2 \) and \( \sigma_1 > \sigma_2 \) to create a trade-off between expected return and risk.
Two Risky Assets

If $w$ is the fraction of initial wealth allocated to asset 1 and $1 - w$ is the fraction of initial wealth allocated to asset 2, the random return $\tilde{r}_P$ on the portfolio is

$$\tilde{r}_P = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

and the expected return $\mu_P$ on the portfolio is

$$\mu_P = E[w\tilde{r}_1 + (1 - w)\tilde{r}_2]$$

$$= wE(\tilde{r}_1) + (1 - w)E(\tilde{r}_2)$$

$$= w\mu_1 + (1 - w)\mu_2$$
Two Risky Assets

\[ \mu_P = w\mu_1 + (1 - w)\mu_2 \]

The expected return on the portfolio is a weighted average of the expected returns on the individual assets.

Since \( \mu_1 > \mu_2 \), \( \mu_P \) can range from \( \mu_2 \) up to \( \mu_1 \) as \( w \) increases from zero to one. Even higher (or lower) expected returns are possible if short selling is allowed.
Two Risky Assets

We calculate the variance of the random portfolio return

\[ \tilde{r}_P = w \tilde{r}_1 + (1 - w) \tilde{r}_2 \]

\[
\sigma_P^2 = E[(\tilde{r}_P - \mu_P)^2] \\
= E([w \tilde{r}_1 + (1 - w) \tilde{r}_2 - w \mu_1 - (1 - w) \mu_2]^2) \\
= E([w(\tilde{r}_1 - \mu_1) + (1 - w)(\tilde{r}_2 - \mu_2)]^2) \\
= E[w^2(\tilde{r}_1 - \mu_1)^2 + (1 - w)^2(\tilde{r}_2 - \mu_2)^2] \\
+ 2w(1 - w)(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2) \\
= w^2E[(\tilde{r}_1 - \mu_1)^2] + (1 - w)^2E[(\tilde{r}_2 - \mu_2)^2] \\
+ 2w(1 - w)E[(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]
\]
Covariance and correlation coefficient

In probability theory, the covariance between two random variables $X_1$ and $X_2$ is defined as

$$\sigma(X_1, X_2) = E([X_1 - E(X_1)][X_2 - E(X_2)])$$

and the correlation between $X_1$ and $X_2$ is defined as

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$
Covariance and correlation coefficient

\[ \sigma(X_1, X_2) = E([X_1 - E(X_1)][X_2 - E(X_2)]) \]

The covariance is

- positive if \( X_1 - E(X_1) \) and \( X_2 - E(X_2) \) tend to have the same sign
- negative if \( X_1 - E(X_1) \) and \( X_2 - E(X_2) \) tend to have opposite signs
- zero if \( X_1 - E(X_1) \) and \( X_2 - E(X_2) \) show no tendency to have the same or opposite signs.
Covariance and correlation coefficient

Mathematically, therefore, the covariance

\[ \sigma(X_1, X_2) = E([X_1 - E(X_1)][X_2 - E(X_2)]) \]

measures the extent to which the two random variables tend to move together.

Economically, buying two assets with returns that are imperfectly, and especially, negatively correlated is like buying insurance: one return will be high when the other is low and vice versa, reducing the overall risk of the portfolio.
Covariance and correlation coefficient

The correlation

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

has the same sign as the covariance, and is therefore also a measure of co-movement.

But “scaling” the covariance by the two standard deviations makes the correlation range between $-1$ and $1$: 
Two Risky Assets

Hence

\[ \sigma_P^2 = w^2 E[(\tilde{r}_1 - \mu_1)^2] + (1 - w)^2 E[(\tilde{r}_2 - \mu_2)^2] \]
\[ + 2w(1 - w)E[(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)] \]
\[ = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_{12} \]
\[ = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12} \]

where

\[ \sigma_{12} = \text{the covariance between } \tilde{r}_1 \text{ and } \tilde{r}_2 \]
\[ \rho_{12} = \text{the correlation between } \tilde{r}_1 \text{ and } \tilde{r}_2 \]
Two Risky Assets

This is the source of the gains from diversification: the expected portfolio return

\[ \mu_P = w \mu_1 + (1 - w) \mu_2 \]

is a weighted average of the expected returns on the individual asset returns, but the standard deviation of the portfolio return

\[ \sigma_P = \left[ w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12} \right]^{1/2} \]

is not a weighted average of the standard deviations of the returns on the individual assets and can be reduced by choosing a mix of assets \((0 < w < 1)\) when \(\rho_{12}\) is less than one and, especially, when \(\rho_{12}\) is negative.
Mean Variance Portfolio (MVP)

$$\sigma_P^2 = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}$$

We can minimize the portfolio variance by setting the first derivative equal to zero:

$$\frac{d\sigma_P^2}{dw} = 2w\sigma_1^2 - 2\sigma_2^2 + 2w\sigma_2^2 + 2\sigma_{12} - 4w\sigma_{12} = 0$$

and solve for $w^*$

$$w^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$
Two Risky Assets: $\rho_{12} = 1$

To see more specifically how this works, start with the case where $\rho_{12} = 1$ so that the individual asset returns are perfectly correlated. This is the one case in which there are no gains from diversification. With $\rho_{12} = 1$

\[
\begin{align*}
\mu_P &= w\mu_1 + (1 - w)\mu_2 \\
\sigma_P &= \left[ w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12} \right]^{1/2} \\
&= \left[ w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2 \right]^{1/2} \\
&= ([w\sigma_1 + (1 - w)\sigma_2]^2)^{1/2} \\
&= w\sigma_1 + (1 - w)\sigma_2
\end{align*}
\]

In this special case, the standard deviation of the return on the portfolio is a weighted average of the standard deviations of the returns on the individual assets.
Two Risky Assets: $\rho_{12} = 1$

When $\rho_{12} = 1$, so that individual asset returns are perfectly correlated, there are no gains from diversification.
Two Risky Assets: $\rho_{12} = 1$

To show that $P_1 P_2$ is a straight line: no matter what percentage of wealth $w$ we choose to invest in $X$ the trade-off between expected value and standard deviation is constant.

\[
\text{Slope} = \frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P/dw}{d\sigma_P/dw} = \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}
\]
Two Risky Assets: $\rho_{12} = 1$

Theorem

In the case of two risky assets with perfectly positively correlated returns ($\rho_{12} = 1$), the efficient frontier is linear; in that extreme case the two assets are essentially identical, there is no gain from diversification.
Two Risky Assets: $\rho_{12} = -1$

Next, let's consider the opposite extreme, in which $\rho_{12} = -1$ so that the individual asset returns are perfectly, but negatively, correlated:

\[
\sigma_P = \left[ w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12} \right]^{1/2}
\]

\[
= \left[ w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 - 2w(1 - w)\sigma_1 \sigma_2 \right]^{1/2}
\]

\[
= \left[ w \sigma_1 - (1 - w) \sigma_2 \right]^2 \right]^{1/2}
\]

\[
= \pm \left[ w \sigma_1 - (1 - w) \sigma_2 \right]
\]

In this special case, the setting

\[
w^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}
\]

\[
= \frac{\sigma_2}{\sigma_1 + \sigma_2}
\]

creates a “synthetic” risk free portfolio (at $w^*$, $\sigma_P = 0$)!
Two Risky Assets: $\rho_{12} = -1$

When $\rho_{12} = -1$, so that individual asset returns are perfectly, but negatively correlated, risk can be eliminated via diversification.
Two Risky Assets: $\rho_{12} = 1$

To show that $P_2 P_{\text{mvp}}$ and $P_{\text{mvp}} P_1$ are linear: the slope is invariant to changes in percentage of an investor’s portfolio invested in $X$

\[
\text{Slope } P_2 P_{\text{mvp}} = \frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P}{dw} \frac{d\sigma_P}{dw} = \frac{\mu_1 - \mu_2}{\sigma_1 + \sigma_2} > 0
\]

\[
\text{Slope } P_{\text{mvp}} P_1 = \frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P}{dw} \frac{d\sigma_P}{dw} = \frac{\mu_1 - \mu_2}{-(\sigma_1 + \sigma_2)} < 0
\]
Two Risky Assets: $\rho_{12} = -1$

**Theorem**

*If the two risky assets have returns that are perfectly negatively correlated ($\rho_{12} = -1$), the minimum variance portfolio is risk free while the frontier is linear.*

If one of the two assets is risk free, then the efficient frontier is a straight line originating on the vertical axis at the level of the risk-free return.
In the absence of a short sales restriction, the overall portfolio can be made riskier than the riskiest among the existing assets; in other words, it can be made riskier than the one risky asset and it must be that the efficient frontier is projected to the right of the \((\mu_2, \sigma_2)\) point.
Two Risky Assets: $-1 < \rho_{12} < 1$

\[
\begin{align*}
\mu_P &= w\mu_1 + (1 - w)\mu_2 \\
\sigma_P &= [w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}]^{1/2}
\end{align*}
\]

In all intermediate cases, there will still be gains from diversification. These gains will become stronger as $\rho_{12}$ declines from 1 to -1.
Minimum Variance Frontier is the locus of risk and return combinations offered by portfolios of risky assets that yield the minimum variance for a given rate of return.

In general, the MVF is convex, because it is bounded by the triangle ABC.
Two Risky Assets: $-1 < \rho_{12} < 1$

As $\rho_{12}$ decreases from 0.5 to 0, -0.5, -0.75, the gains from diversification strengthen.
Two Risky Assets: \(-1 < \rho_{12} < 1\)

**Theorem**

*In the case of two risky assets with imperfectly correlated returns* \((-1 < \rho_{12} < 1)\), *the standard deviation of the portfolio is necessarily smaller than it would be if the two component assets were perfectly correlated:*

\[
\sigma_P < w\sigma_1 + (1 - w)\sigma_2
\]

The smaller the correlation (further away from +1), the more to the left is the MVF.
Example

Let $R_1$ and $R_2$ be the returns for two securities with $E(R_1) = 0.03$ and $E(R_2) = 0.08$, $Var(R_1) = 0.02$, $Var(R_2) = 0.05$ and $cov(R_1, R_2) = -0.01$. Assuming that the two securities above are the only investments available, plot the set of feasible mean-variance combinations of return.

<table>
<thead>
<tr>
<th>% in 1</th>
<th>% in 2</th>
<th>$E(r_P)$</th>
<th>var($r_P$)</th>
<th>$\sigma_P$</th>
</tr>
</thead>
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<tr>
<td>150</td>
<td>-50</td>
<td>0.5%</td>
<td>7.25%</td>
<td>26.93%</td>
</tr>
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<td>0</td>
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<td>125</td>
<td>10.5</td>
<td>13.25</td>
<td>36.4</td>
</tr>
</tbody>
</table>
Example

Figure: Mean-Variance Frontier

- (0%, 100%)
- (25%, 75%)
- (50%, 50%)
- (75%, 25%)
- (100%, 0%)
- (150%, -50%)
- (-50%, 150%)
Example

If we want to minimize risk, how much of our portfolio will we invest in security 1?

\[ w^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = 0.67 \]

If we put two-thirds into asset 1, the portfolio’s standard deviation is

\[ E(R_p) = 0.67 \times 0.03 + 0.33 \times 0.08 = 4.65\% \]
\[ var(R_P) = 0.67^2 \times 0.02 + 0.33^2 \times 0.05 + 2 \times 0.67 \times 0.33 \times (−0.01) \]
\[ = 0.01 \]

The MVP is represented by the intersection of the dashed lines in the figure.
Two Risky Assets and More?

\[
\begin{align*}
\mu_P &= w\mu_1 + (1 - w)\mu_2 \\
\sigma_P &= \left[w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}\right]^{1/2}
\end{align*}
\]

In the case with two risky assets, the choice of \( w \) simultaneously determines \( \mu_P \) and \( \sigma_P \). Because a portfolio is also an asset fully defined by its expected return, its standard deviation, and its correlation with other existing assets or portfolios; The previous analysis with 2 assets is more general than it appears as it can easily be repeated with one of the two assets being a portfolio.

But with more than two risky assets, the portfolio problem takes on an added dimension, since then we can ask: how can we select \( w_1, w_2, \ldots, w_N \) to minimize \( \sigma_P \) for any given choice of \( \mu_P \)?
Consider two portfolios, A and B, with expected returns $\mu_A$ and $\mu_B$ and standard deviations $\sigma_A$ and $\sigma_B$.

Recall that portfolio A is said to exhibit mean-variance dominance over portfolio B if either

$$\mu_A > \mu_B \quad \text{and} \quad \sigma_A \leq \sigma_B$$

$$\mu_A \geq \mu_B \quad \text{and} \quad \sigma_A < \sigma_B$$

Hence, choosing portfolio shares to minimize variance for a given mean will allow us to characterize the efficient frontier: the set of all portfolios that are not mean-variance dominated by any other portfolio.

This is a useful intermediate step in modern portfolio theory, since investors with mean-variance utility will only choose portfolios on the efficient frontier.
Three Risky Assets

With three assets, for example, an investor can choose

\[
\begin{align*}
    w_1 &= \text{share of initial wealth allocated to asset 1} \\
    w_2 &= \text{share of initial wealth allocated to asset 2} \\
    1 - w_1 - w_2 &= \text{share of initial wealth allocated to asset 3}
\end{align*}
\]

Given the choices of \( w_1 \) and \( w_2 \)

\[
\begin{align*}
    \hat{r}_P &= w_1 \hat{r}_1 + w_2 \hat{r}_2 + (1 - w_1 - w_2) \hat{r}_3 \\
    \mu_P &= w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 \\
    \sigma^2_P &= w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + (1 - w_1 - w_2)^2 \sigma^2_3 \\
    &\quad + 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} + 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
    &\quad + 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23}
\end{align*}
\]
Three Risky Assets

Our problem is to solve

$$\min_{w_1, w_2} \sigma_P^2$$

$$s.t. \quad \mu_P = \bar{\mu}$$

for a given value of $\bar{\mu}$.

But since we are more used to solving constrained maximization problems, consider the reformulated, but equivalent, problem:

$$\max_{w_1, w_2} -\sigma_P^2$$

$$s.t. \quad \mu_P = \bar{\mu}$$
Three Risky Assets

Set up the Lagrangian, using the expressions for $\mu_P$ and $\sigma_P$ derived previously:

\[
L = -\left[ w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 \right. \\
+ 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} + 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
+ 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \\
+ \left. \lambda \left[ w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu} \right] \right]
\]
Three Risky Assets

F.O.C. for $w_1$

$$0 = -2w_1^* \sigma_1^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_2^* \sigma_1 \sigma_2 \rho_{12} - 2(1 - w_1^* - w_2^*) \sigma_1 \sigma_3 \rho_{13} + 2w_1^* \sigma_1 \sigma_3 \rho_{13} + 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_1 - \lambda^* \mu_3$$

F.O.C. for $w_2$

$$0 = -2w_2^* \sigma_2^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_1^* \sigma_1 \sigma_2 \rho_{12} + 2w_1^* \sigma_1 \sigma_3 \rho_{13} - 2(1 - w_1^* - w_2^*) \sigma_2 \sigma_3 \rho_{23} + 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_2 - \lambda^* \mu_3$$

B.C.

$$w_1^* \mu_1 + w_2^* \mu_2 + (1 - w_1^* - w_2^*) \mu_3 = \bar{\mu}$$
Three Risky Assets

The two first-order conditions and the constraint form a system of three equations in the three unknowns: \( w_1^*, w_2^* \) and \( \lambda^* \).

Moreover, the equations are linear in the unknowns \( w_1^*, w_2^* \) and \( \lambda^* \). Given specific values for \( \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3, \rho_{12}, \rho_{13}, \rho_{23}, \) and \( \bar{\mu} \) they can be solved quite easily.
In linear algebra, a vector is just a column of numbers. With $N \geq 3$ assets, you can organize the portfolio shares and expected returns into vectors:

$$ w = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_N \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \mu_N \end{bmatrix} $$

where

$$ w_1 + w_2 + ... + w_N = 1 $$

Also in linear algebra, the transpose of a vector just reorganizes the column as a row; for example:

$$ w^\prime = [w_1, w_2, ..., w_N] $$
Meanwhile, the variances and covariances can be organized into a matrix — a collection of rows and columns:

\[
\Sigma = 
\begin{bmatrix}
\sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \ldots & \sigma_1\sigma_N\rho_{1N} \\
\sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \ldots & \sigma_2\sigma_N\rho_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_1\sigma_N\rho_{1N} & \sigma_2\sigma_N\rho_{2N} & \ldots & \sigma_N^2 \\
\end{bmatrix}
\]
Linear Algebra

Using the rules from linear algebra for multiplying vectors and matrices, the expected return on any portfolio with shares in the vector $w$ is

$$\mu'w$$

and the variance of the random return on the portfolio is

$$w'\Sigma w$$

Hence, the problem of minimizing the variance for a given mean can be written compactly as

$$\max_w -w'\Sigma w \quad \text{s.t.} \quad \mu'w = \bar{\mu} \quad \text{and} \quad l'w = 1$$

where $l$ is a vector of $N$ ones.
Linear Algebra

\[
\max_w -w'\Sigma w \quad \text{s.t.} \quad \mu'w = \bar{\mu} \quad \text{and} \quad l'w = 1
\]

Problems of this form are called quadratic programming problems and can be solved very quickly on a computer even when the number of assets \( N \) is large.

We can also add more constraints, such as \( w_i \geq 0 \), ruling out short sales.
Three Risky Assets

Going back to the case with three assets, once the optimal shares \( w_1^* \) and \( w_2^* \) have been found, the minimized standard deviation can be computed using the general formula

\[
\sigma_P^2 = \sigma_1^2 w_1^2 + \sigma_2^2 w_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 \\
+ 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12} + 2 w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
+ 2 w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23}
\]

Doing this for various values of \( \bar{\mu} \) allows us to trace out the minimum variance frontier.
Adding assets shifts the minimum variance frontier to the left, as opportunities for diversification are enhanced.
However, the minimum variance frontier retains its sideways parabolic shape.
The **minimum variance frontier** traces out the minimized variance or standard deviation for each required mean return.
But portfolio A exhibits mean-variance dominance over portfolio B, since it offers a higher expected return with the same standard deviation.
Hence, the **efficient frontier** extends only along the top arm of the minimum variance frontier.
Recall that either of two sets of assumptions will imply that indifference curves in this $\mu - \sigma$ diagram slope upward and are convex:

- Investors have vN-M expected utility with quadratic Bernoulli utility functions
- Asset returns are normally distributed and investors have vN-M expected utility with increasing and concave Bernoulli utility functions
Portfolios along $U_1$ are mean-variance dominated by others. Portfolios along $U_3$ are infeasible. Portfolio $P^*$, located where $U_2$ is tangent to the efficient frontier, is optimal.
The Optimal Portfolio Choice

Each indifference curve maps out all combinations of risk and return that provide us with the same utility. The slope of indifference curve indicates the **marginal rate of substitution (MRS)** between our preference for risk and return, which is subjective.

The efficient frontier shows the tradeoff between risk and return, the slope of which indicates the **marginal rate of transformation (MRT)** offered by MVF.

An important feature of the optimal portfolio that we choose to maximize our utility is that the subjective MRS is exactly equal to the objectively determined MRT between risk and return.
Investor B is less risk averse than investor A. Different investors face the same assessment of the return and risk offered by risky assets, they may hold different portfolios. But all optimal portfolios are along the efficient frontier. Thus, the mean-variance utility hypothesis built into Modern Portfolio Theory implies that all investors choose optimal portfolios along the efficient frontier.
Efficient Frontier with 1 risky asset and 1 risk-free asset

So far, however, our analysis has assumed that there are only risky assets. An additional, quite striking, result emerges when we add a risk free asset to the mix.

Consider, therefore, the larger portfolio formed when an investor allocates the fraction $w$ of his or her initial wealth to a risky asset or to a smaller portfolio of risky assets and the remaining fraction $1 - w$ to a risk free asset with return $r_f$.

If the risky part of this portfolio has random return $\tilde{r}$, expected return $\mu_r = E(\tilde{r})$, and variance $\sigma_r^2 = E[(\tilde{r} - \mu_r)^2]$ then the larger portfolio has random return $\tilde{r}_P = w\tilde{r} + (1 - w)r_f$ with expected return

$$\mu_P = E[w\tilde{r} + (1 - w)r_f] = w\mu_r + (1 - w)r_f$$

and variance

$$\sigma_P^2 = E[(\tilde{r}_P - \mu_P)^2]$$

$$= E[w\tilde{r} + (1 - w)r_f - w\mu_r - (1 - w)r_f]^2$$

$$= E[w(\tilde{r} - \mu_r)]^2 = w^2 \sigma_r^2$$
The expression for the portfolio’s variance

\[ \sigma_P^2 = w^2 \sigma_r^2 \]

implies

\[ \sigma_P = w \sigma_r \]

Hence

\[ w = \frac{\sigma_P}{\sigma_r} \]

Hence, with \( \sigma_r \) given, a larger share of wealth \( w \) allocated to risky assets is associated with a higher standard deviation \( \sigma_P \) for the larger portfolio.
But the expression for the portfolio’s expected return

\[ \mu_P = w \mu_r + (1 - w) r_f \]

indicates that so long as \( \mu_r > r_f \), a higher value of \( w \) will yield a higher expected return as well.

What is the trade-off between risk \( \sigma_P \) and expected return \( \mu_P \) of the mix of risky and riskless assets?
To see, substitute

$$w = \frac{\sigma_P}{\sigma_r}$$

into

$$\mu_P = w \mu_r + (1 - w)r_f$$

to obtain

$$\mu_P = \frac{\sigma_P}{\sigma_r} \mu_r + (1 - \frac{\sigma_P}{\sigma_r})r_f$$

$$= r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P$$
Efficient Frontier with 1 risky asset and 1 risk-free asset

\[
\mu_P = r_f + \left( \frac{\mu_r - r_f}{\sigma_r} \right) \sigma_P
\]

shows that for portfolios of risky and riskless assets:

- The relationship between \( \sigma_P \) and \( \mu_P \) is linear.
- The slope of the linear relationship is given by the Sharpe ratio, defined here as the “expected excess return” offered by the risky components of the portfolio divided by the standard deviation of the return on that risky component:

\[
\frac{\mu_r - r_f}{\sigma_r}
\]
It is usually assumed that the rate of return on the risk-free asset is equal to the borrowing and lending rate in the economy.
Hence, any investor can combine the risk free asset with risky portfolio A to achieve a combination of expected return and standard deviation along the red line.
However, any investor with mean-variance utility will prefer some combination of the risk free asset and risky portfolio B to all combinations of the risk free asset and risky portfolio A.
And **all investors with mean-variance utility** will prefer some combination of the risk free asset and risky portfolio T to any other portfolio.
Theorem

With $N$ risky assets and a risk-free one, the efficient frontier is a straight line.

We call $T$ the **tangency portfolio**. As before, if we allow short position in the risk-free asset, the efficient frontier extends beyond $T$. 
We now ask whether and how the MVF construction can be put to service to inform actual portfolio practice: one result is surprising.

- The optimal portfolio is naturally defined as that portfolio maximizing the investor’s (mean-variance) utility; That portfolio for which he is able to reach the highest indifference curve in MV space;

- Such curves will be increasing and convex from the origin; They are increasing because additional risk needs to be compensated by higher means; They are convex if and only if the investor is characterized by increasing absolute risk aversion (IARA), which is the case under MV preferences, as we have claimed
Investor B is less risk averse than investor A. But both choose same combination of the “tangency portfolio” T and the risk free asset.
Note that the tangency portfolio $T$ can be identified as the portfolio along the efficient frontier of risky assets that has the highest Sharpe ratio.
Separation Theorem

**Theorem**

*Any risk averse investor, independently of her risk aversion, will diversify between a risky (tangency portfolio) fund and the riskless asset.*

- It is natural to realize that if there is a risk-free asset, then all tangency points must lie on the same efficient frontier, irrespective of the coefficient of risk aversion of each specific investor.

- Let there be two investors sharing the same perceptions as to expected returns, variances, and return correlations but differing in their willingness to take risks.
Separation Theorem

**Theorem**

*Any risk averse investor, independently of her risk aversion, will diversify between a risky (tangency portfolio) fund and the riskless asset.*

- The relevant efficient frontier will be identical for these two investors, although their optimal portfolios will be represented by different points on the same line.
- With differently shaped indifference curves the tangency points must differ.
Separation Theorem

It is a fact that our two investors will invest in the same two funds, the risk-free asset on the one hand, and the risky portfolio \((T)\) identified by the tangency point between the straight line originating from the vertical axis and the efficient frontier.

- It implies that the optimal portfolio of risky assets can be identified separately from the knowledge of the risk preference of an investor.
- Notice that this important result applies regardless of the (possibly non normal) probability distributions of returns representing the subjective expectations of the particular investor.
Separation Theorem

This is the **two-fund theorem** or **separation theorem** implied by Modern Portfolio Theory.

Equity mutual fund managers can all focus on building the unique portfolio that lies along the efficient frontier of risky assets and has the highest Sharpe ratio.

Each individual investor can then tailor his or her own portfolio by choosing the combination of the riskless assets and the risky mutual fund that best suits his or her own aversion to risk.
We’ve already considered one shortcoming of the MPT: its mean-variance utility hypothesis must rest on one of two more basic assumptions.

Either utility must be quadratic or asset returns must be normal.
Pros and Cons of MPT

A second problem involves the estimation or “calibration” of the model’s parameters.

With $N$ risky assets, the vector $\mu$ of expected returns contains $N$ elements and the matrix $\Sigma$ of variances and covariances contains $N(N+1)/2$ unique elements. When $N = 100$, for example, there are $100 + (100 \times 101)/2 = 5150$ parameters to estimate!

And to use data from the past to estimate these parameters, one has to assume that past averages and correlations are a reliable guide to the future.
Pros and Cons of MPT

On the other hand, the MPT teaches us a very important lesson about how individual assets with imperfectly, and especially negatively, correlated returns can be combined into a diversified portfolio to reduce risk.

And the MPT’s separation theorem suggests that a retirement savings plan that allows participants to choose between a money market mutual fund and a well-diversified equity fund is fully optimal under certain circumstances and perhaps close enough to optimal more generally.
Pros and Cons of MPT

Finally, our first equilibrium model of asset pricing, the Capital Asset Pricing Model, builds directly on the foundations provided by Modern Portfolio Theory.
Summary

- There is no contradiction between the way in which an economist looks at portfolio problems and what is typically done in practice in finance.
- We have defined mean-variance preferences and analyzed their microeconomic foundations, which may be exact (quadratic utility, jointly normally distributed returns) or approximated (Taylor).
- We have built the minimum variance and mean-variance efficient frontiers for a variety of cases, with and without constrains.
- We have examine how a risk-averse, IARA investor should be optimizing her portfolio with and without a riskless asset.
- The separation, or two-fund theorem emerged rather naturally from our work; we have discussed its implications for the asset management industry.
- We developed mean-variance closed-form asset allocation formulas.